# TECHNICAL NOTE

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# A CORRELATION FUNCTION DEFINITION OF SPECTRAL RESOLUTION

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SUMMARY

The process of spectral resolution is examined for each of three intrinsically different optical systems: the prism, the interferometric spectrometer, and the recently developed multifilter method. It is suggested that a generalized definition of resolution must apply to all such systems. One approach which seems sufficiently general in scope to be applicable to all possible spectrally dispersing systems is the use of the correlation functions of the outputs. This method is illustrated for the prism and interferometric cases. The classical Rayleigh criterion is extended to the interferometric system, as a specific application.

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## INTRODUCTION

By the term spectral resolution is implied the smallest increment of wavelength that a spectrally dispersing system can resolve. The criteria used to determine spectral resolution are fairly clear in the case of the usual dispersing systems such as the prism, grating, etc. But they have not been adequately defined for more unusual systems such as the interferometric spectrometer and the multifilter system. In the latter systems dispersion is accomplished mathematically rather than physically. It would seem, therefore, that a generalized definition of spectral resolution is desirable.

In the present treatment, the prism and interferometric spectrometer are used to illustrate a criterion based on the correlation functions of the system outputs; this approach seems to be general enough for application to all possible spectrally dispersing systems.

## COMPARISON OF THE SYSTEMS

In the case of the prism, ignoring diffraction, an incident plane wave represented by the input function of wavelength, $f(\lambda)$, is mapped into an output function of angle, $g(\beta)$, given by

$$g(\beta) = \int K(\lambda, \beta) f(\lambda) d\lambda,$$

where the kernel $K$ represents the transfer function of the prism. The prism thus maps the $\lambda$ space into the $\beta$ space. Here the spectral dispersion is arrived at by using two delta functions as input functions, $\delta(\lambda - \lambda_1)$ and $\delta(\lambda - \lambda_2)$. The output then consists correspondingly of $\delta(\beta - \beta_1)$ and $\delta(\beta - \beta_2)$, and the dispersion is given by

$$\frac{\beta_2 - \beta_1}{\lambda_2 - \lambda_1} = \frac{\Delta\beta}{\Delta\lambda} .$$



The dispersion depends, of course, on the specific kernel $K$. Since $K$ for the prism gives a one-to-one correspondence between $\lambda$ and $\beta$, it is not necessary to transform inversely to find $f(\lambda)$.

If diffraction is now assumed present in the kernel $K$, the latter does not map delta functions into delta functions but into broadened functions (although the mapping is still closely one-to-one), and the usual definition of resolution is arrived at by considering the dispersion coupled with the diffraction.

In interferometric spectroscopy (References 1, 2) an input function $f(\lambda)$ is mapped into an output function of path difference, $g(x)$, given by

$$g(x) = \int K(\lambda, x) f(\lambda) d\lambda.$$

Here $K$ does not merely perform a one-to-one mapping, but is more complex, being equal to $\cos 2\pi x/\lambda$. This results in two essential differences from the prism case. First, it is not possible to define dispersion simply; and second, it is necessary to make the inverse transform. Since the range of $x$ involved in the integration of the inverse transform is necessarily finite, analogous to the finite aperture of the prism, "diffraction" appears again and limits the resolution obtained. The direct transform here may be considered to produce both dispersion and diffraction, while the inverse transform produces further diffraction.

In the multifilter method (Reference 3) of spectral analysis, ideally an input function $f(\lambda)$ is mapped by a transmission function $K(\lambda, t)$ into an output function $g(t)$, where $t$ is a continuous variable such as time:

$$g(t) = \int K(\lambda, t) f(\lambda) d\lambda.$$

In practice, the integral equation is replaced by a matrix equation since $K(\lambda, t)$ is constructed from a finite number of filters. The resolution-determining factors present in the interferometer method – i.e., the nature of $K$ and the inverse transformation – are both present here and are accompanied by a third factor, the discreteness of the system.

## CORRELATION FUNCTION AS A MEASURE OF RESOLUTION

It would seem that a general definition of spectral resolution should take all three systems into account. It might be arrived at by the operational approach mentioned in connection with the prism, i.e., using two delta functions as input functions and finding the smallest separation between them that the given system will resolve. Thus the direct transform of a system will give, corresponding to two delta functions $\delta(\lambda - \lambda_1)$, and $\delta(\lambda - \lambda_2)$,

$$\int K(\lambda, t)\, \delta(\lambda - \lambda_1)\, d\lambda = g_1(t),$$

$$\int K(\lambda, t)\, \delta(\lambda - \lambda_2)\, d\lambda = g_2(t).$$

If the inverse transforms of $g_1(t)$ and $g_2(t)$ are denoted by $g_1'(\lambda)$ and $g_2'(\lambda)$, their correlation function

$$\int_{-\infty}^{\infty} g_1'(\lambda)\, g_2'(\lambda)\, d\lambda$$

may be a useful measure of the congruence between the two system outputs, and therefore a useful measure of the resolution.

The prism and interferometric systems will now be discussed to illustrate how the Rayleigh criterion of resolution may be carried over to the general case by means of the correlation function.

### Case 1 – Prism

The output function of the prism for an input delta function $\delta(\lambda - \lambda_1)$ is

$$\left(\frac{\sin \dfrac{2\pi d \sin \theta}{\lambda}}{\dfrac{2\pi d \sin \theta}{\lambda}}\right)^2 = \frac{\sin^2 x}{x^2} = f(x),$$

where

$$x = \frac{2\pi d \sin \theta}{\lambda},$$

and $d$ is the aperture.

The correlation function is given by (Reference 4)

$$\phi(x) = 2\pi \int_{-\infty}^{\infty} E(y)\, e^{ixy}\, dy,$$

where $E(y)$ is the square of the Fourier transform of $f(x)$, a triangular function here:

$$E(y) = \begin{cases} \left[\frac{1}{2}\left(1 - \frac{y}{2}\right)\right]^2 & (0 < y < 2), \\ \left[\frac{1}{2}\left(1 + \frac{y}{2}\right)\right]^2 & (-2 < y < 0), \\ 0 & \begin{pmatrix} -\infty < y \le -2 \\ 2 \le y < \infty \end{pmatrix} . \end{cases}$$

The resulting correlation function is

$$\phi_P(x) = \frac{2\pi(2x - \sin 2x)}{4x^3} .$$

For the Rayleigh separation $\theta_R = \lambda/2d$, $\phi_P$ has the value

$$\phi_P(x_R) \approx \frac{1}{\pi} .$$

This may be normalized relative to the value of $\phi_P(0) = 2\pi/3$, giving

$$\frac{\phi_P(\theta_R)}{\phi_P(0)} \approx \frac{3}{2\pi^2} \approx 0.152 .$$

Two wavelengths will be resolved when their diffraction patterns have a correlation less than this value.

### Case 2 – Interferometric Spectrometer

For an input delta function $\delta(\lambda - \lambda_1)$, the interferometric system gives the broadening function (References 1 and 2)

$$f(x) = \frac{\sin x}{x} \qquad \left[x = 2\pi l\left(\frac{1}{\lambda_1} - \frac{1}{\lambda}\right)\right],$$

$l$ being the path difference. The normalized correlation function is

$$\frac{\phi_i(x)}{\phi_i(0)} = \frac{\sin x}{x} ,$$

which has the value 0.152, corresponding to the value found for the correlation of the Rayleigh separation in the prism system, for $x \approx 0.95\pi$, so that

$$\frac{1}{\lambda_1} - \frac{1}{\lambda} \approx \frac{0.95}{2l} \approx \frac{\lambda - \lambda_1}{\lambda^2} .$$

This gives, for the resolving power,

$$\frac{\lambda}{\lambda - \lambda_1} \approx \frac{2l}{0.95\lambda} .$$

In this case a resolving power is arrived at because the "dispersion" of the system is essentially known.

## CONCLUDING REMARKS

The application of this approach to the multifilter system is more difficult, but would yield a quantity for a given transmission matrix which could be compared with the Rayleigh definition in the same manner as is done for the interferometer.

The Rayleigh criterion is used here only for purposes of comparison. It may not be the best standard for correlation, since output functions of various systems will, in general, bear little resemblance to the diffraction function, on which the Rayleigh criterion was based.

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